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DOES THE INTERACTION PICTURE EXIST?

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An argument, based on what would normally be regarded as a reasonable set of assumptions, suggests that in Relativistic Quantum Field Theory, there is no Interaction Picture. The proof requires a reformulation of field theory which does not assume this, and it is found that this reformulation needs no renormalisation, suggesting that the false assumption of the existence of the interaction picture is responsible for the usual divergences.

1. Introduction

In his book *Lectures on Quantum Field Theory* Dirac suggests, on the basis of certain problems of formulating field theory, that the Schrödinger Picture does not exist in a Relativistic Quantum Field Theory. I suppose that it is natural that the notion that the Interaction or *Dirac* picture would not exist would not cross his mind, but a set of arguments may be presented which lead inexorably to this conclusion. We begin with a simpleminded argument, and then proceed to back this up with a more substantial analysis wherein the case of a self-interacting scalar field theory is analysed. The simpleminded argument runs as follows–

If the interaction picture exists, then there is a unitary transformation $U(t) = e^{iHt}e^{-iH_0t}$ which relates states of the free theory to those of the interacting theory, assuming that they were the same at $t = 0$. Thus

$$\Phi(t, \mathbf{x}) = U(t)\Phi_0(t, \mathbf{x})U(t)^\dagger \tag{1}$$

gives the interacting field operator in terms of the free field operator. In relativistic notation this is

$$\Phi(x) = U(n \cdot x)\Phi_0(x)U(n \cdot x)^\dagger \tag{2}$$

where $n^a = (1, 0, 0, 0)$. However, if the theory is properly relativistic, then the relationship cannot be true in one frame without being true in every other frame, so we must assume that it is true for all n satisfying $n_0 > 0$, $n^2 = 1$. Thus any pair of points separated by a spacelike interval must have the same unitary transformation which relates free and interacting fields since we can always draw a hyperplane through these points whose normal is a future-pointing timelike vector. Now *any* pair of points will have points separated from both by a spacelike interval, so we are forced to conclude that $U(t)$ is universal– i.e. that it is actually independent of t. So $U(t) = U(0) = 1$ and the theory is just a free theory. The construction of the operators $U(t)$ relies on being able to separate the Hamiltonian into "free" and "interacting" parts. This decomposition, it is proposed, is not possible in a Relativistic Quantum Field Theory.

2. The assumptions

i The states of a physical system form a linear vector space $\mathcal{V}$ over the complex numbers $\mathcal{C}$, and this is equipped with a sesquilinear, positive-definite inner product.

ii There exists a self-adjoint linear map $\Phi : \mathcal{M} \otimes \mathcal{V} \to \mathcal{V}$, called the "field operator", where $\mathcal{M}$ is Minkowski space. All states and operators may be defined from this field operator.

iii There exists a representation of the identity-connected Poincaré group on $\mathcal{V}$, which preserves the inner product.

iv There exists a Poincaré-invariant state, $|0\rangle$, called the vacuum.

v All the eigenvalues of the translation generators, or four-momentum operator P_a lie on or within the forward light-cone.

vi A pair of field operators will either commute or anticommute when the spacetime points they refer to are separated by a spacelike interval. The (anti)commutators of fields referring to the same spacetime point are always c-numbers.

Since this would generally be regarded as a reasonable way of defining a Hermitian, scalar, self-interacting field theory– together with the specification of the equation of motion– no apology will be made for these.

3. The free scalar field

The existence of the Interaction Picture is an *assumption* in the usual formulation of field theory, and the problem becomes the calculation of $U(t)$ for large values of t. Here we are not assuming that $U(t)$ exists and are thereby forced to derive field theory in a different way. Our first consideration, then, is free field theory, since, as will be seen, our method consists in the reducing of the interacting field into tensor products of free fields. Also, it is desirable to show that the assumptions of §2 are sufficient to determine free field theory entirely, and we do not have to introduce canonical quantisation as an extra ingredient.

We require the free field to give a state which belongs to a unitary irreducible representation of the Poincaré group of mass m and spin zero. It must therefore obey the Klein-Gordon equation

$$(\partial^2 + m^2)\Phi(x) = 0 \tag{3}$$

Consequently, a field defined at any time is linearly related to $\Phi(0, \mathbf{x})$ and $\dot{\Phi}(0, \mathbf{x})$, the field and its time derivative at $t = 0$. The (anti)commutators of these, by assumption (vi), are c-numbers, so the (anti)commutator

$$[\Phi(x), \Phi(x')]_\pm$$

is always a c-number. Forming the Fourier transform

$$\Phi(p) = (2\pi)^{-4}\int d^4x\; e^{-ip\cdot x}\Phi(x) \tag{4}$$

it follows that

$$[\Phi(p), \Phi(p')]_\pm \tag{5}$$

is a c-number also. Now

$$[P_\mu, \Phi(x)] = -i\partial_\mu\Phi(x) \tag{6}$$

from which we find

$$[P_\mu, \Phi(p)] = p_\mu\Phi(p). \tag{7}$$

Forming the commutator of P_μ with the (anti)commutator (5) we then see that, using a Jacobi identity, this is zero unless $p + p' = 0$. So the c-number must have a factor $\delta(p + p')$. It must also have a factor $\delta(p^2 - m^2)$ since $\Phi(p)$ vanishes unless $p^2 = m^2$. This leads us to a most general form

$$[\Phi(p), \Phi(p')]_\pm = [a.\theta(p_0) + b.\theta(-p_0)]\; \delta(p^2 - m^2)\; \delta(p + p') \tag{8}$$

where a and b are constants. Returning to configuration space, we see that

$$[\Phi(x), \Phi(x')]_\pm = a.f(x - x') + b.f(x' - x) \tag{9}$$

$$\text{where} \qquad f(x) = \int \frac{d^3\mathbf{p}}{2p^0}\; e^{ip\cdot x}\bigg|_{p^0=\sqrt{\mathbf{p}^2+m^2}} \tag{10}$$

For a spacelike vector, $f(x) = f(-x)$ (but this is not true in general), so to satisfy assumption (vi), we need $b = -a$. Thus

$$[\Phi(p), \Phi(p')]_{\pm} = a.\epsilon(p_0)\delta(p^2 - m^2)\delta(p + p'). \tag{11}$$

Exchanging p' and p and multiplying by (± 1) leaves the (anti)commutator invariant. Hence $a = \mp a$ – i.e. the system must be quantised with commutators, which is of course in accordance with the spin-statistics theorem. Now $\Phi(p) = \Phi^\dagger(-p)$, since the field is Hermitian, so we can write

$$[\Phi^\dagger(p), \Phi(p')] = -a.\epsilon(p_0)\delta(p^2 - m^2)\delta(p - p'). \tag{12}$$

Forming the vacuum expectation value of this quantity, we then see that if $p_0 > 0$ and $p_0' > 0$, then the second contribution of the commutator, by assumption (v), is zero (otherwise we would have a state of negative energy). The first part is positive or zero, which tells us that the constant a is real and negative. We shall choose $a = -1$, since any other choice could be realised by a simple rescaling of the fields. Thus, our assumptions lead to

$$[\Phi^\dagger(p), \Phi(p')] = \epsilon(p_0)\delta(p^2 - m^2)\delta(p - p'), \tag{13}$$

which is unique, apart from the scaling factor.

In configuration space, then, we have

$$[\Phi(x), \Phi(x')] = i(2\pi)^3\Delta(x - x') \tag{14}$$

where $$\Delta(x) = -i\int \frac{d^4p}{(2\pi)^3}\, \delta(p^2 - m^2)\epsilon(p_0)\, e^{-ip.x} \tag{15}$$

is the usual commutator function.

Hence the theory gives the same equal time commutators as the usual method (apart from the $(2\pi)^3$ normalisation, which is simply a different choice of normalisation convention).

The Fourier transform fields relate to the annihilation and creation operators through

$$\Phi(p) = \delta(p^2 - m^2)[\theta(p_0)a^\dagger(\mathbf{p}) + \theta(-p_0)a(-\mathbf{p})]. \tag{16}$$

The annihilation operators annihilate the vacuum on account of assumption (v), i.e. because otherwise we would have a negative energy state: we do not, therefore, need to "prove" this on the basis of positive definiteness of the Hamiltonian as constructed by the canonical quantisation method.

4. The interacting field

The equation of motion of Φ^3 theory is

$$(\partial^2 + m^2)\Phi(x) = -\lambda\Phi(x)^2. \tag{17}$$

In terms of the four-dimensional Fourier transforms defined by (4), this is

$$(p^2 - m^2)\Phi(p) = \lambda\int d^4q\; \Phi(q)\Phi(p - q) \tag{18}$$

We may solve this for the case where the physically significant matrix elements are continuous and infinitely differentiable functions of λ, since we can then make the expansion

$$\Phi(p[,\lambda]) = \Phi_0(p) + \lambda\Phi_1(p) + \lambda^2\Phi_2(p) + \cdots \tag{19}$$

The equation (18) then becomes an infinite set of equations, one for each power of λ. These are

$$\begin{aligned}
(p^2 - m^2)\Phi_0(p) &= 0 \\
(p^2 - m^2)\Phi_1(p) &= \int d^4q\ \Phi_0(q)\Phi_0(p-q) \\
(p^2 - m^2)\Phi_2(p) &= \int d^4q\ (\Phi_0(q)\Phi_1(p-q) + \Phi_1(q)\Phi_0(p-q)) \\
&\vdots \\
(p^2 - m^2)\Phi_r(p) &= \int d^4q\ \sum_{i=0}^{r-1} \Phi_i(q)\Phi_{r-i-1}(p-q) \\
&\vdots
\end{aligned} \tag{20}$$

$\Phi_0(p)$, clearly, is just a free field, and the higher-order terms are reducible, directly or indirectly, to tensor products of Φ_0's.

Our system must obey assumption (vi), which constrains the possible form of the equation of motion. We now temporarily abandon the particular study of Φ^3 theory to see what these constraints are in general, taking with us only the results that the system has a characteristic coupling λ, such that we may expand the field about $\lambda = 0$, and that Φ_0 is a free field, with the higher-order terms being formed from tensor products of these.

The free-field equal-time commutators are

$$\begin{aligned}
[\Phi(\mathbf{x},t), \Phi(\mathbf{x}',t)] &= 0 \\
[\Phi(\mathbf{x},t), \dot{\Phi}(\mathbf{x}',t)] &= i(2\pi)^3\delta(\mathbf{x}-\mathbf{x}') \\
[\dot{\Phi}(\mathbf{x},t), \dot{\Phi}(\mathbf{x}',t)] &= 0
\end{aligned} \tag{21}$$

which may be rewritten in terms of the Fourier transforms thus:

$$\int_{-\infty}^{\infty} d\nu\ \{1, \nu, \nu^2\}\ [\Phi(r+\nu n), \Phi(q-r-\nu n)] = \{0, -\delta(q), 0\} \tag{22}$$

where $n^a = (1,0,0,0)$, but r and q are arbitrary four-vectors except that $r \cdot n = 0$. Since the theory is covariant, this holds for all n satisfying $n_0 > 0$, $n^2 = 1$, which is equivalent to saying that the commutators (21) hold for any choice of spacelike hyperplane.

In the presence of interactions we assume that these commutators still hold. We assume also, then, that the c-numbers on the right-hand side are independent of the coupling constant. If they were not, then it would only be to the extent of a λ-dependent scale factor, which could be removed by a rescaling of the fields. The constraint on the higher-order fields is then

$$\int_{-\infty}^{\infty} d\nu\ \{1, \nu, \nu^2\} \sum_{i=0}^{k} [\Phi_i(r+\nu n), \Phi_{k-i}(q-r-\nu n)] = 0 \tag{23}$$

where $k > 0$. This can be solved. Let us do this for the first-order field, where

$$\int_{-\infty}^{\infty} d\nu\ \{1, \nu, \nu^2\}\ ([\Phi_0(r+\nu n), \Phi_1(q-r-\nu n)] + [\Phi_1(r+\nu n), \Phi_0(q-r-\nu n)]) = 0 \tag{24}$$

It follows from this that

$$\begin{gathered}
\int_{-\infty}^{\infty} d\nu \left((\nu \pm R)((\nu-\xi)^2 - S^2) - (\nu - \xi[\pm]S)(\nu^2 - R^2) \right) \times \\
([\Phi_0(r+\nu n), \Phi_1(q-r-\nu n)] + [\Phi_1(r+\nu n), \Phi_0(q-r-\nu n)]) = 0.
\end{gathered} \tag{25}$$

since the cubic terms cancel. Here $R = \sqrt{m^2 - r^2}$, $\xi = n \cdot q$, $s = q - r - \xi n$ and $S = \sqrt{m^2 - s^2}$. The notation "[±]" means an independent choice of signs to the other "±". Each of the quadratic factors cancels one of the Φ_0's, so we can write (25) as

$$\int_{-\infty}^{\infty} d\nu \left((\nu \pm R)((\nu - \xi)^2 - S^2)[\Phi_0(r + \nu n), \Phi_1(q - r - \nu n)] \quad - \right.$$
$$\left. (\nu - \xi[\pm]S)(\nu^2 - R^2)[\Phi_1(r + \nu n), \Phi_0(q - r - \nu n)] \right) = 0 \tag{26}$$

The commutator of Φ_0 with a functional of Φ_0 can be reduced to a sum of terms involving individual commutators. The general formula is

$$[\Phi_0(p), S[\Phi_0]] = -\frac{\delta S}{\delta \Phi_0(-p)} \delta(p^2 - m^2)\epsilon(p_0) \tag{27}$$

where the functional derivative is defined by

$$\frac{\delta S[\phi]}{\delta \phi(q)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(S[\phi(p) + \epsilon.\delta(p - q)] - S[\phi(p)]). \tag{28}$$

$$\text{Thus} \quad \int_{-\infty}^{\infty} d\nu \left((\nu \pm R)((\nu - \xi)^2 - S^2) \frac{\delta \Phi_1(q - r - \nu n)}{\delta \Phi_0(-r - \nu n)} \delta((r + \nu n)^2 - m^2)\epsilon(r_0 + \nu n_0) \quad + \right.$$
$$\left. (\nu - \xi[\pm]S)(\nu^2 - R^2) \frac{\delta \Phi_1(r + \nu n)}{\delta \Phi_0(-q + r + \nu n)} \delta((q - r - \nu n)^2 - m^2)\epsilon(q_0 - r_0 - \nu n_0) \right) = 0 \tag{29}$$

So

$$((q - r_\pm)^2 - m^2) \frac{\delta \Phi_1(q - r_\pm)}{\delta \Phi_0(-r_\pm)} = ((q - s_{[\mp]})^2 - m^2) \frac{\delta \Phi_1(q - s_{[\mp]})}{\delta \Phi_0(-s_{[\mp]})}, \tag{30}$$

where $r_\pm = r \pm Rn$ and $s_\pm = s \pm Sn$.

The set of three vectors, q, one of $r_\pm$, and one of $s_\pm$ are independent. Hence both sides of (30) depend on q only, from which we conclude that

$$(p^2 - m^2) \frac{\delta \Phi_1(p)}{\delta \Phi_0(-q)} = C(p + q) \tag{31}$$

where $C(p + q)$ is some operator. Now, the most general possible form for $(p^2 - m^2)\Phi_i(p)$ is a sum of terms of the form

$$\int d^4p_1 d^4p_2 \cdots d^4p_r \; M_r(p - p_1 - p_2 - \cdots - p_r, p_1, p_2, \cdots, p_r)\Phi_0(p_1)\Phi_0(p_2) \cdots \Phi_0(p_r) \tag{32}$$

where M_r are c-number-valued functions. Applying the functional derivative, we see that eqn. (31) requires that each M_r is independent of the p_i's except as they appear in the first slot. But, forming the commutator with P_μ we obtain the result that the p_i's must add up to p, which means that there must be a momentum-conserving delta function in the expression. The freedom is thus reduced to a single constant which we will still call M_r, and the contribution to $(p^2 - m^2)\Phi_i(p)$ is

$$M_r \int d^4p_1 d^4p_2 \cdots d^4p_r \delta(p - p_1 - p_2 - \cdots - p_r)\Phi_0(p_1)\Phi_0(p_2) \cdots \Phi_0(p_r). \tag{33}$$

This is a *local* construction, since in configuration space this is

$$(\partial^2 + m^2)\Phi_1(x) = -\sum_{r=2}^{\infty} M_r \Phi_0(x)^r \tag{34}$$

The higher-order fields can now be derived straightforwardly. To do this we note that

$$\tilde{\Phi}_k(p) = (p^2 - m^2)\Phi_k(p) = \sum_r M_r \int d^4p_1 d^4p_2 \cdots d^4p_r \delta(p - p_1 - p_2 - \cdots - p_r)$$
$$\sum_{i_1 i_2 \cdots i_r} \delta_{k-i, i_1+i_2+\cdots+i_r} \Phi_{i_1}(p_1)\Phi_{i_2}(p_2)\cdots\Phi_{i_r}(p_r) \tag{35}$$

is a formula that works for $k = 1$. To show that it works also for $k > 1$, we use the equation

$$\int_{-\infty}^{\infty} d\nu \sum_{m=0}^{k} (\nu \pm R)[\Phi_m(r + \nu n), \tilde{\Phi}_{k-m}(q - r - \nu n)] \quad -$$
$$(\nu - \xi[\pm]S)[\tilde{\Phi}_m(r + \nu n), \Phi_{k-m}(q - r - \nu n)] = 0, \tag{36}$$

which follows from (23). Substituting the expression (35) for each appearance of $\tilde{\Phi}$, and expanding the commutators, we reduce this to terms which are zero provided that the spacelike commutators of lower order hold. This establishes (35) as an expression which gives the higher-order fields. We have not, however, proved that the solution is unique. Examining (23) we see that a higher-order field is necessarily arbitrary to the extent of adding a solution of the kind Φ_1. Allowing in such a solution at higher order has the effect of leading to a series of terms at orders which are multiples of this, constructed in the same way as the series (35) which is seeded by order λ fields. The best way of demonstrating this is to note that the relation (35) for giving higher-order fields is equivalent to saying that the system has an equation of motion

$$(p^2 - m^2)\Phi(p) = \lambda \sum_r M_r \int d^4p_1 d^4p_2 \cdots d^4p_r \delta(p - p_1 - p_2 - \cdots - p_r)\Phi(p_1)\Phi(p_2)\cdots\Phi(p_r). \tag{37}$$

If Φ_1 type solutions are introduced at (say) order k, then this is represented by an order λ^k term in the coupling. Since they may be introduced at any higher order, it then follows that the most general solution is

$$(p^2 - m^2)\Phi(p) = \sum_r M_r(\lambda) \int d^4p_1 d^4p_2 \cdots d^4p_r \delta(p - p_1 - p_2 - \cdots - p_r)\Phi(p_1)\Phi(p_2)\cdots\Phi(p_r) \tag{38}$$

where $M_r(\lambda)$ are a set of polynomials in λ, such that $M_r(0) = 0$. So, in configuration space,

$$(\partial^2 + m^2)\Phi(x) = -\sum_r M_r(\lambda)\Phi(x)^r \tag{39}$$

is the equation of motion that expresses the most general solution to the spacelike commutativity requirement.

The points to note about this result are, firstly, that the equation of motion should be local is what we would expect from spacelike commutativity, since this is rooted in the notion that information cannot propagate faster than the speed of light. Secondly, no derivatives are allowed in the coupling. Our method then shows that this is based on spacelike commutativity rather than "renormalisability".

It is well known that the infinities of quantum field theory are a consequence of the locality of the equations of motion. If one takes the view, as the author does, that pathological divergences are signals of a fundamental inconsistency in the theory, then one could say that the problem is that assumptions (v) and (vi) do not peacefully coexist. However, it is possible to formulate a much cleaner treatment of infinities than is usual if one uses this method, avoiding, for the time being at least, the precise specification of the equations of motion. The procedure is as follows.

The infinities arise from the terms Φ_1 in the power series expansion, upwards. Examining (33), and imagining that there is a vacuum state which lies to the right of it, we obtain infinite quantities when we commute the negative-energy parts of the Φ_0's past positive-energy parts of Φ_0's standing to the right to annihilate the vacuum. These infinities are readily removable. If we order the terms in the product so that the annihilation parts always stand to the right, then this is equivalent to subtracting infinite amounts of local field products of the same kind, but of lower order, so that, in principle at least, the normal ordered product still has the form of a solution of the spacelike commutativity requirement. The proof of the validity of this procedure is that, as may easily be verified, eqn. (31) still holds. Thus the normal ordering of the interaction at first order does not present the slightest problem. The removal of infinities in the fields of order two and higher is, however, a little less rigorous. A higher-order field of order k first appears, and is defined by, the extremal terms in the summation (36). These terms are the pair which appears alone in the definition of Φ_1, and in exact analogy to this case we can arrange the expression as

$$\begin{aligned} ((q-r_\pm)^2 - m^2)\frac{\delta\Phi_n(q-r_\pm)}{\delta\Phi_0(-r_\pm)} \quad - \quad \hat{F}_n(q, r_\pm) = \\ ((q-s_{[\mp]})^2 - m^2)\frac{\delta\Phi_n(q-s_{[\mp]})}{\delta\Phi_0(-s_{[\mp]})} \quad - \quad \hat{F}_n(q, s_{[\mp]}) \end{aligned} \tag{40}$$

where $\hat{F}_n$ is an operator-valued function. If we put both sides equal to zero, then we may integrate to get (35). In general, both sides equal a function of q only, and the inclusion of this would lead to the introduction of a Φ_1 type solution at this order. Now if $\hat{F}_n$ has its component fields in normal order then we can integrate the expression in such a way that no infinities would be introduced into the higher-order field. However this is not in general the case. Putting $\hat{F}_n$ into normal order requires a series of infinite quantities to be subtracted from the original expression. We need to do this, so we justify it by noting that the terms to be subtracted are infinite regardless of the values of $r_\pm$, and thus are identical to the corresponding terms in terms of $s_{[\pm]}$. As noted earlier, the problem is one of the incompatibility of assumptions (v) and (vi), so the best we can do is to say that *if* assumptions (v) and (vi) are compatible, then this is the series of higher-order fields we obtain. No other pathological divergences appear to arise in this formalism.

5. The non-existence of the Interaction Picture

We are now in a position to supply a strong negative answer to the original question. We work with Φ^3 theory, with infinities suitably removed. We consider the matrix element

$$\langle 0|\Phi(\mathbf{x},t)\Phi(\mathbf{x}',t)\Phi(\mathbf{x}'',t)|0\rangle. \tag{41}$$

If the interaction picture exists, then, assuming that the vacuum is unitarily invariant (which is usual, apart from a possible infinite phase, which would not in any case upset the argument here), this is equal to the same matrix element of the *free* fields:

$$\langle 0|\Phi_0(\mathbf{x},t)\Phi_0(\mathbf{x}',t)\Phi_0(\mathbf{x}'',t)|0\rangle, \tag{42}$$

which is necessarily zero. However, using the power series method developed earlier, we find an order λ contribution

$$\begin{aligned} &\lambda(\langle 0|\Phi_1(\mathbf{x},t)\Phi_0(\mathbf{x}',t)\Phi_0(\mathbf{x}'',t)|0\rangle + \langle 0|\Phi_0(\mathbf{x},t)\Phi_1(\mathbf{x}',t)\Phi_0(\mathbf{x}'',t)|0\rangle + \langle 0|\Phi_0(\mathbf{x},t)\Phi_0(\mathbf{x}',t)\Phi_1(\mathbf{x}'',t)|0\rangle) \\ &\quad = \int d^4p\; d^4p'\; d^4p''\; e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{x}'+\mathbf{p}''\cdot\mathbf{x}'')}\delta(p+p'+p'').2\lambda. \\ &\qquad \Big(\frac{1}{p^2-m^2}\theta(p_0')\delta(p'^2-m^2)\theta(p_0'')\delta(p''^2-m^2) \quad + \\ &\qquad \theta(-p_0)\delta(p^2-m^2)\frac{1}{p'^2-m^2}\theta(p_0'')\delta(p''^2-m^2) \quad + \\ &\qquad \theta(-p_0)\delta(p^2-m^2)\theta(-p_0')\delta(p'^2-m^2)\frac{1}{p''^2-m^2}\Big) \\ &= -\frac{\lambda}{2}\int d^3\mathbf{p}\; d^3\mathbf{p}'\; d^3\mathbf{p}''\; e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{x}'+\mathbf{p}''\cdot\mathbf{x}'')}\delta(\mathbf{p}+\mathbf{p}'+\mathbf{p}'')\frac{1}{EE'E''(E+E'+E'')} \end{aligned} \tag{43}$$

where $E = \sqrt{\mathbf{p}^2+m^2}$, etc. This does not vanish, hence there is no Interaction Picture. One may readily confirm that other choices of equations of motion and matrix elements lead us to the same conclusion.

6. Life without the Interaction Picture

The result that we have derived, if true, has very wide implications for quantum field theory. It means, in fact, that the whole Feynman-Dyson methodology must be discarded in favour of a more axiomatic approach, such as the one developed here.

However, despite its inconsistencies, the Feynman-Dyson method obviously works well for many physical systems so we must show that the method used here will substantially reproduce these results. In fact, the correspondence with the orthodoxy is very natural, as we shall now see.

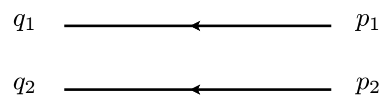
The lowest-order "physical" process in Φ^3 theory is two-body scattering. The relevant matrix element is:

$$\langle 0|\Phi(t,-\mathbf{q}_1)\Phi(t,-\mathbf{q}_2)\Phi(0,\mathbf{p}_2)\Phi(0,\mathbf{p}_1)|0\rangle = \int dq_1^0 dq_2^0 dp_1^0 dp_2^0\, e^{-i(q_1^0+q_2^0)t}\, \langle 0|\Phi(-q_1)\Phi(-q_2)\Phi(p_2)\Phi(p_1)|0\rangle. \quad (44)$$

The contributions to $\langle 0|\Phi(-q_1)\Phi(-q_2)\Phi(p_2)\Phi(p_1)|0\rangle$ are readily obtained with the power series method. There is also a very natural graphical representation:

$\langle 0|\Phi_0\Phi_0\Phi_0\Phi_0|0\rangle$:

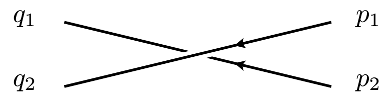
a.



$$\theta(p_1^0)\delta(p_1^2-m^2)\delta(p_1-q_1)\theta(p_2^0)\delta(p_2^2-m^2)\delta(p_2-q_2)$$

$\langle 0|\Phi_0\Phi_0\Phi_0\Phi_0|0\rangle$:

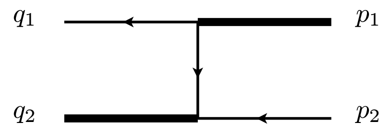
b.



$$\theta(p_1^0)\delta(p_1^2-m^2)\delta(p_1-q_2)\theta(p_2^0)\delta(p_2^2-m^2)\delta(p_2-q_1)$$

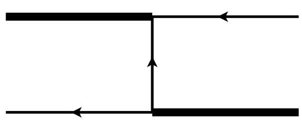
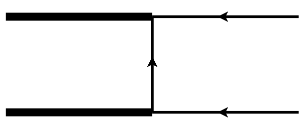
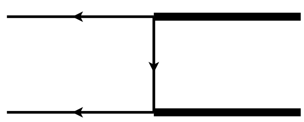
Two Φ_0's and two Φ_1's:

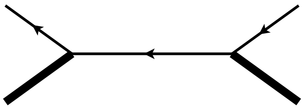
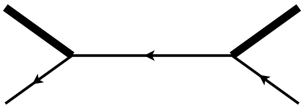
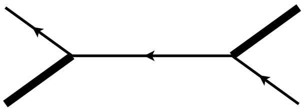
c.

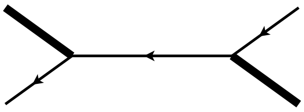
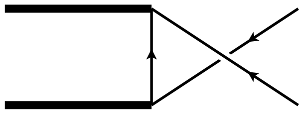
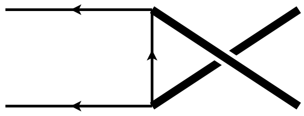


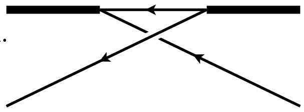
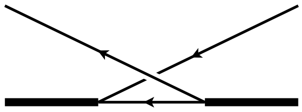
$$\lambda^2\theta(q_1^0)\delta(q_1^2-m^2)\frac{1}{p_1^2-m^2}\theta(p_1^0-q_1^0)\delta((p_1-q_1)^2-m^2)\frac{1}{q_2^2-m^2}\theta(p_2^0)\delta(p_2^2-m^2)\delta(q_1+q_2-p_1-p_2)$$

Also, correspondingly, there are the graphs:

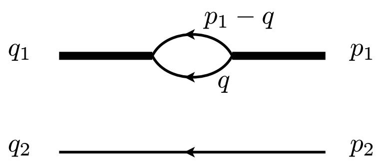
d.  + e.  + f.  +

g.  + h.  + i.  +

j.  + k.  + l.  +

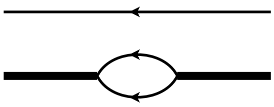
m.  + n.  +

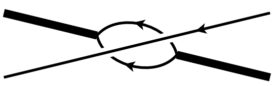
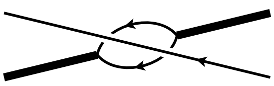
There are four "self-energy" graphs also:

o. 

$$\frac{\lambda^2}{q_1^2 - m^2}\int d^4q\,\theta(q^0)\delta(q^2 - m^2)\theta(p_1^0 - q^0)\delta((p_1 - q)^2 - m^2)\frac{1}{p_1^2 - m^2}\delta(p_1 - q_1)\theta(p_2^0)\delta(p_2^2 - m^2)\delta(p_2 - q_2).$$

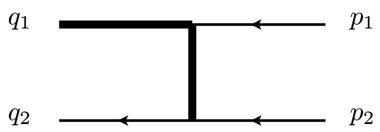
Plus expressions corresponding to the graphs

p.  +

q.  + r. 

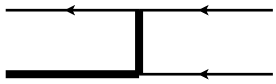
Three Φ_0's and one Φ_2, this completing the set for the two-body scattering amplitude up to second order:

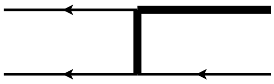
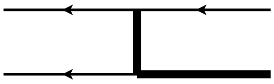
a'.



$$\lambda^2\theta(p_1^0)\delta(p_1^2-m^2)\frac{1}{q_1^2-m^2}\frac{1}{(p_1-q_1)^2-m^2}\theta(p_2^0)\delta(p_2^2-m^2)\theta(q_2^0)\delta(q_2^2-m^2)\delta(q_1+q_2-p_1-p_2),$$

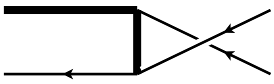
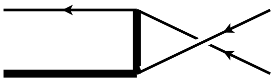
similarly, we have graphs

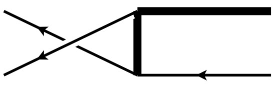
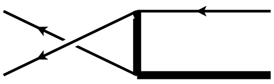
b'.  +

c'.  + d'.  +

e'.  + f'.  +

g'.  + h'.  +

i'.  + j'.  +

k'.  + l'. 

The thin lines, which we call propagators, represent terms obtained when the negative-energy part of a Φ_0 is commuted past a positive-energy Φ_0 standing to the right. Hence the factor $\theta(p^0)\delta(p^2-m^2)$ for these. The heavy lines, which we call proliferators, on the other hand represent the terms associated with the reduction of higher-order fields to lower-order ones, such as in (35). It is easy to convince oneself of the graph rules that proliferators (i) cannot form loops of (just) proliferators and (ii) cannot join external lines to each other by a path consisting solely of proliferators. The loops of propagators are finite, since these are just phase space integrals over a finite phase space. The infinities presumed removed earlier would appear as proliferator trees with propagators attached at both ends to the same tree.

Examination of these graphs reveals that the last twelve can exhibit a property not possessed by the others, namely the possibility of *resonance*: i.e. the proliferators connecting to external lines may become infinite given appropriate values of the four-momenta. This is as distinct from the proliferator in the middle which either has spacelike momenta, or has $p^2 \geq 4m^2$ when it is a sum (the proliferators in the first group of second-order graphs always satisfy one or other of these conditions). Thus the amplitude approaches infinity as the particles approach being on-shell. The approximation we will make is that this pole dominates the expression. Consider the expression for graph a': the contribution to the matrix element (44) is

$$\frac{\lambda^2}{2E(\mathbf{p}_1)2E(\mathbf{p}_2)2E(\mathbf{q}_2)}\frac{1}{q_1^{0\,2}-E(\mathbf{q}_1)^2}\frac{1}{(p_1-q_1)^2-m^2}\;e^{-i(E(\mathbf{p}_1)+E(\mathbf{p}_2))t}\delta(\mathbf{q}_1+\mathbf{q}_2-\mathbf{p}_1-\mathbf{p}_2) \tag{45}$$

where $E(\mathbf{p}) = \sqrt{\mathbf{p}^2+m^2}$ etc., $p_1^\mu = (E(\mathbf{p}_1),\mathbf{p}_1)$, $p_2^\mu = (E(\mathbf{p}_2),\mathbf{p}_2)$, $q_2^\mu = (E(\mathbf{q}_2),\mathbf{q}_2)$, and $q_1^\mu = (E(\mathbf{p}_1)+E(\mathbf{p}_2)-E(\mathbf{q}_2),\mathbf{q}_1)$.

Evidently the pole domination assumption is tantamount to saying that $(q_1^0 - E(\mathbf{q}_1))^{-1} = (E(\mathbf{p}_1)+E(\mathbf{p}_2)-E(\mathbf{q}_1)-E(\mathbf{q}_2))^{-1}$ is the pole that dominates, so that we can put q_1 on shell in the rest of the expression. This gives us

$$\begin{aligned}&\frac{\lambda^2}{2E(\mathbf{p}_1)2E(\mathbf{p}_2)2E(\mathbf{q}_1)2E(\mathbf{q}_2)}\frac{1}{(p_1-q_1)^2-m^2}\;e^{-(E(\mathbf{p}_1)+E(\mathbf{p}_2))t}\times\\&\qquad\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_2)-E(\mathbf{q}_1)-E(\mathbf{q}_2)}\delta(\mathbf{q}_1+\mathbf{q}_2-\mathbf{p}_1-\mathbf{p}_2)\end{aligned} \tag{46}$$

where all the momenta are on shell now. The other eleven graphs give pole contributions which are derived in exactly the same way. Doing this, we find that the zeroth-order for two-body scattering, plus the pole parts in second-order, gives

$$\begin{aligned}\langle 0|\Phi(t,-\mathbf{q}_1)\Phi(t,-\mathbf{q}_2)\Phi(0,\mathbf{p}_2)\Phi(0,\mathbf{p}_1)|0\rangle \quad &= \\ \frac{e^{-i(E(\mathbf{p}_1)+E(\mathbf{p}_2))t}}{2E(\mathbf{p}_1)2E(\mathbf{p}_2)}(\delta(\mathbf{p}_1-\mathbf{q}_1)\delta(\mathbf{p}_2-\mathbf{q}_2)+\delta(\mathbf{p}_1-\mathbf{q}_2)\delta(\mathbf{p}_2-\mathbf{q}_1)) \quad &+ \\ \frac{2\lambda^2}{2E(\mathbf{p}_1)2E(\mathbf{p}_2)2E(\mathbf{q}_1)2E(\mathbf{q}_2)}\frac{e^{-i(E(\mathbf{q}_1)+E(\mathbf{q}_2))t}-e^{-i(E(\mathbf{p}_1)+E(\mathbf{p}_2))t}}{E(\mathbf{q}_1)+E(\mathbf{q}_2)-E(\mathbf{p}_1)-E(\mathbf{p}_2)}&\times \\ \left(\frac{1}{(p_1-q_1)^2-m^2}+\frac{1}{(p_1+p_2)^2-m^2}+\frac{1}{(p_1-q_2)^2-m^2}\right)\delta(\mathbf{q}_1+\mathbf{q}_2-\mathbf{p}_1-\mathbf{p}_2).&\end{aligned} \tag{47}$$

Within this approximation, the interaction picture exists. This is most readily seen by comparing with

$$\begin{aligned}\langle\mathbf{q}_1\mathbf{q}_2|e^{iHt}|\mathbf{p}_2\mathbf{p}_1\rangle = e^{i(E(\mathbf{p}_1)+E(\mathbf{p}_2))t}\delta(\mathbf{p}_1-\mathbf{q}_1)\delta(\mathbf{p}_2-\mathbf{q}_2) \quad +&\\ \langle\mathbf{q}_1\mathbf{q}_2|V|\mathbf{p}_2\mathbf{p}_1\rangle\frac{e^{i(E(\mathbf{q}_1)+E(\mathbf{q}_2))t}-e^{i(E(\mathbf{p}_1)+E(\mathbf{p}_2))t}}{E(\mathbf{q}_1)+E(\mathbf{q}_2)-E(\mathbf{p}_1)-E(\mathbf{p}_2)}+\mathcal{O}(V^2)&\end{aligned} \tag{48}$$

which is the scattering amplitude up to first order, for a two-particle system in ordinary quantum mechanics, where $H = H_0 + V$, and input and output particle states are eigenstates of H_0. In this case, where the particles are identical, we need to insist that the wavefunctions multiplying the states are symmetric with respect to particle exchange. The differential cross-section for particle scattering for such a system is given by

$$d\sigma = \frac{1}{v}d^3\mathbf{q}_1 d^3\mathbf{q}_2|\langle\mathbf{q}_1\mathbf{q}_2||V||\mathbf{p}_2\mathbf{p}_1\rangle|^2(2\pi)^4\delta^4(q_1+q_2-p_1-p_2) \tag{49}$$

where v is the velocity of one particle beam in a frame in which the other is at rest. The reduced matrix element $\langle\ ||\cdots||\ \rangle$ differs from the other in that the three-momentum conservation delta function has been extracted.

The differing sign of t is irrelevant, so making a shift

$$\Phi(.,\mathbf{p}_2)\Phi(.,\mathbf{p}_1)|0\rangle \to \frac{1}{\sqrt{2}}\sqrt{2E(\mathbf{p}_2)}\sqrt{2E(\mathbf{p}_1)}\Phi(.,\mathbf{p}_2)\Phi(.,\mathbf{p}_1)|0\rangle \tag{50}$$

we see that the system is like the above with a reduced matrix element

$$\langle \mathbf{q}_1\mathbf{q}_2||V||\mathbf{p}_2\mathbf{p}_1\rangle = \frac{\lambda^2}{\sqrt{2E(\mathbf{p}_1)2E(\mathbf{p}_2)2E(\mathbf{q}_1)2E(\mathbf{q}_2)}} \times \left(\frac{1}{(p_1-q_1)^2-m^2}+\frac{1}{(p_1+p_2)^2-m^2}+\frac{1}{(p_1-q_2)^2-m^2}\right) \tag{51}$$

so the scattering cross section is

$$d\sigma = \frac{1}{v}\frac{d^3\mathbf{q}_1}{2E(\mathbf{q}_1)}\frac{d^3\mathbf{q}_2}{2E(\mathbf{q}_2)}\frac{1}{2E(\mathbf{p}_1)}\frac{1}{2E(\mathbf{p}_2)}(2\pi)^4\delta(q_1+q_2-p_1-p_2) \times \lambda^4\left[\frac{1}{(p_1-q_1)^2-m^2}+\frac{1}{(p_1+p_2)^2-m^2}+\frac{1}{(p_1-q_2)^2-m^2}\right]^2 \tag{52}$$

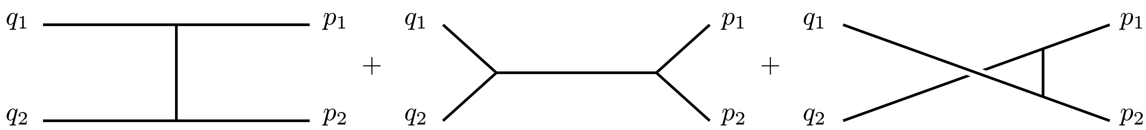
In this formulation we have chosen normalisations which avoid factors of $(2\pi)^3$ appearing as much as possible. This means that we need to make the substitutions

$$\begin{aligned}\Phi &\to (2\pi)^{3/2}\Phi\\ \lambda &\to (2\pi)^{-3/2}\lambda\end{aligned} \tag{53}$$

to compare with the usual formulation. We obtain

$$d\sigma = \frac{1}{v}\frac{d^3\mathbf{q}_1}{(2\pi)^3 2E(\mathbf{q}_1)}\frac{d^3\mathbf{q}_2}{(2\pi)^3 2E(\mathbf{q}_2)}\frac{1}{2E(\mathbf{p}_1)}\frac{1}{2E(\mathbf{p}_2)}\lambda^4 \times \left[\frac{1}{(p_1-q_1)^2-m^2}+\frac{1}{(p_1+p_2)^2-m^2}+\frac{1}{(p_1-q_2)^2-m^2}\right]^2(2\pi)^4\delta(q_1+q_2-p_1-p_2) \tag{54}$$

This agrees with the expression obtained from considering the Feynman graphs



apart from a numerical factor: in the Feynman graph approach this result is obtained with an equation of motion $(\partial^2+m^2)\Phi = -\lambda\Phi^2/2!$ whereas we have obtained it with an equation of motion $(\partial^2+m^2)\Phi = -\lambda\Phi^2$.

7. Conclusion and outlook

We have seen that it is possible to reproduce the result of Feynman graph analysis for two-body scattering in a scalar theory using a formalism that does not assume the existence of the interaction picture. In doing so we were able to obtain our result by much more mathematically acceptable methods than usual: specifically, the "renormalisation" procedure was not required. It is easy to see how our methods extend to reproduce the results of any analysis involving just "tree" Feynman graphs.

Having done this the next task is to extend our formalism to the analysis of real processes. A preliminary investigation has shown that one can formulate a "Salam-Weinberg" theory of Weak and Electromagnetic interactions quite straightforwardly. Gauge theories are forbidden in our formalism: non-Abelian ones specifically because they involve derivative couplings, but more generally because the method does not permit massless particles of spin greater than one half (by an argument entirely analogous to that given in §4 one may establish that vector-vector couplings that involve derivatives belong to fields which do not commute for spacelike intervals; vector fields may not be totally massless since there then would be graphs– in fact those corresponding to (o) and (p) of §6 – from which we could never abolish gauge dependence). Hence we need to approach the Salam-Weinberg theory *after* the symmetry has been broken, and allow the photon to have a mass, however small. The procedure is then simple: leave out all of the Higgs fields and the vector-vector couplings, and put the masses in directly, by hand. What is known about the Weak force is based on single Boson exchange, and an analysis on the same lines as that in §6 shows us that our results for these processes will then be the same. We do not have to worry about violating "unitarity" because, without an interaction picture, there is no "unitarity" to violate. Also, we do not have to worry about bad terms in the vector propagator, since the "propagator" here is a proliferator, and this is just the Green function associated with the equation of motion, which is simply $g_{\mu\nu}(q^2 - M_B^2)^{-1}$, where M_B is the Boson mass.

Thus the Weak interaction, and scattering processes in Quantum Electrodynamics are easy to model. However the real test is whether we can obtain expressions for the Lamb Shift and the anomalous magnetic moment which agree with experiment as well as the usual method. At the time of writing (June 1986) the situation is as follows: it is impossible to treat systems where the encounter time is large (i.e. bound states, and the classical limit, both of which need to be understood to get these fine corrections) by considering only the low-order graphs. This is easily shown, but if we pretend that this is not a problem we *may* get the correct gross and fine structure of the Hydrogen atom by considering single photon exchange but at the fine structure level, the results depend on interpretation. The only satisfactory answer is based on the study of the appropriate Bethe-Salpeter equation, which, within this formalism, can be written down and (I hope) solved.

References

P.A.M. Dirac, *Lectures in Quantum Field Theory*, Academic Press, N.Y. (1966).

For an account of the Feynman-Dyson approach, including the development of Quantum Electrodynamics and the Salam-Weinberg model, a suitable reference is: C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill (1980).