Stückelberg covariant perturbation theory: *timeline*

1934 Stückelberg: 1st quantized electron, 2nd quantized photon; 4D Fourier transforms of fields; field expansion in coupling; physical interpretation by analysing poles. Compton scattering only.

1950 Källén: 2nd quantized fields; expansion in coupling; all processes, but no cross-sections. No reference to Stückelberg.

1955 Haag: expansion of scalar field in coupling. Attempt to solve commutators.

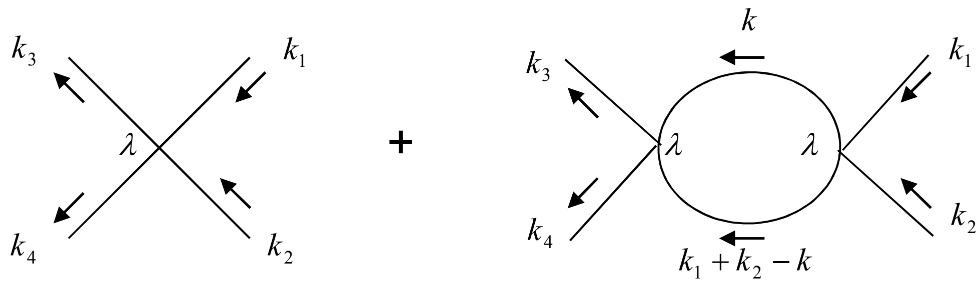
1972 Källén: Earlier work appears in his *Quantum Electrodynamics* text book.

1984-7 CGO: Independent reinvention of Stückelberg work, but all processes and with solving of commutators.

1999 Lacki, Ruegg, Telegdi: review article on Stückelberg work.

Renormalization: why you should be ashamed of yourselves

E.g. Zee, *Quantum Field Theory in a Nutshell*, p. 149.



(Elastic scattering in Φ^4 theory) Loop diagram is a divergent integral, so the theory does not work. End of story!

Zee cuts off the integral with a parameter Λ, leading to the expression

$$-i\lambda + iC\lambda^2 \log\left(\frac{\Lambda^6}{(k_1+k_2)^2(k_1-k_3)^2(k_1-k_4)^2}\right)$$

Yet beside the fact that nothing gave us the right to cut off the integral, there is no requirement for Λ to be a constant – by adjusting the functional form here, one can get *any answer one wants!*

All we know for sure is that naïve Feynman-Dyson perturbation theory does not work!

“Effective” field theory just consists in fitting physical amplitudes with a set of pre-determined functions without any axiomatic basis: the success of Quantum Electrodynamics is just the success in choice of fitting functions and has no basis in quantum field theory!

Can one do better?

I think so ...

The 4D Fourier transform

Any operator-function of the spacetime co-ordinate $x = (ct, \boldsymbol{x})$ will, by definition, have the property that

$$[P_\mu, \Phi(x)] = -i\partial_\mu \Phi(x)$$

The Fourier transform, defined by

$$\Phi(p) = (2\pi)^{-4} \int d^4x\, e^{-ip\cdot x}\, \Phi(x)$$

will therefore obey

$$[P_\mu, \Phi(p)] = p_\mu \Phi(p)$$

$\Phi(p)$ is thus an operator of 4-momentum p, and therefore

$$|\,p\rangle = \Phi(p)|0\rangle$$

is a *state* of 4-momentum p. Note that, thus far, we have said very little about Φ: it could be a free field, an interacting field. It could also be some kind of a composite.

Now: let us consider the specific case of Φ^3 theory. The equation of motion is

$$(\partial^2 + m^2)\,\Phi(x) = -\lambda\,\Phi^2(x)$$

which in momentum space is

$$(p^2 - m^2)\,\Phi(p) = \lambda \int d^4q\, \Phi(q)\,\Phi(p-q)$$

(using the convolution theorem)

The expansion of the field in the coupling

The technique pioneered by Stückelberg, and later, quasi-independently, by Källén. We start with a MacLaurin expansion of the field:

$$\Phi(p) = \Phi_0(p) + \lambda\Phi_1(p) + \lambda^2\Phi_2(p) + \cdots$$

The momentum-space equation of motion then becomes an infinite number of equations, one for each power of λ:

$$(p^2 - m^2)\Phi_0(p) = 0$$

$$(p^2 - m^2)\Phi_1(p) = \int d^4q\,\Phi_0(q)\Phi_0(p-q)$$

$$(p^2 - m^2)\Phi_2(p) = \int d^4q\,\left(\Phi_0(q)\Phi_1(p-q) + \Phi_1(q)\Phi_0(p-q)\right)$$

$$\vdots$$

$$(p^2 - m^2)\Phi_r(p) = \int d^4q\,\sum_{i=0}^{r-1}\Phi_i(q)\Phi_{r-i-1}(p-q)$$

Φ_o is thus a free field, and the higher-order terms are determined from this. The theory is thus *entirely determined from free fields*.

Calculating matrix elements

Fortunately, we know how to handle free fields. In terms of the more-familiar annihilation and creation operators, we can write Φ_0 thus:

$$\Phi_0(p) = \delta(p^2 - m^2)\left[\theta(p_0)a^+(\mathbf{p}) + \theta(-p_0)a(-\mathbf{p})\right]$$

The commutator function is then

$$[\Phi_0(p), \Phi_0(q)] = (2\pi)^{-3}\,\delta(p+q)\,\delta(q^2 - m^2)\,\varepsilon(q^0)$$

... and, in principle at least, we can then calculate any matrix element by inspection, expanding the higher-order fields in terms of Φ_0 and then commuting the negative-energy (annihilation) parts past to annihilate the vacuum. Källén represents these amplitudes using something akin to Feynman graphs; however since there are two kinds of “propagator”, with factors respectively

$$(2\pi)^{-3}\,\theta(p_0)\,\delta(p^2 - m^2) \text{ and } (p^2 - m^2)^{-1}$$

I personally draw the first with a thin line, and the second with a heavy one.

Scattering amplitudes

2-on-2 scattering is represented by the matrix element

$$\left\langle 0 \middle| \Phi(t;\mathbf{p}_3)\Phi(t;\mathbf{p}_4)\Phi(0;\mathbf{p}_1)\Phi(0;\mathbf{p}_2) \middle| 0 \right\rangle$$

where we have Fourier-transformed back just the time components. Consideration of the $<\Phi_0\Phi_2\Phi_0\Phi_0>$ and $<\Phi_0\Phi_0\Phi_2\Phi_0>$ contribution gives expressions of the form

$$\frac{e^{i(E(\mathbf{p_3})+E(\mathbf{p_4}))t} - e^{i(E(\mathbf{p_1})+E(\mathbf{p_2}))t}}{E(\mathbf{p_3})+E(\mathbf{p_4})-E(\mathbf{p_1})-E(\mathbf{p_2})}$$

where $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$

This is approximated with an energy-conservation delta function in the same way as is done with time-dependent perturbation theory in quantum mechanics, and the amplitudes are the same as those obtained from Feynman graphs for the process.

Caveats

1. **Infinities.** Contractions between different expansion trees never diverge. Loops will always be just phase space integrals and so cannot be infinite. However, contractions within an expansion tree are, more often than not, divergent integrals and seem to be unavoidable as long as one is using local field equations. The answer, probably, is just not to use local field equations, but rather "the expansion that would have arisen from a local field equation, but put in normal order".

2. **Haag's theorem is manifest.** The matrix element

$$\langle 0 | \Phi(\mathbf{x},t)\Phi(\mathbf{x}',t)\Phi(\mathbf{x}'',t) | 0 \rangle$$

is non-zero. This vanishes identically for free fields, so no unitary operator $U(t)$ exists which transforms interacting fields into free fields. This is Haag's theorem, and makes it hard to make the correspondence with ordinary quantum mechanics.

Spacelike (anti-)commutativity

The requirement that fields commute or anti-commute at spacelike intervals leads to the spin-statistics theorem for free fields. In momentum space the requirement for a scalar field translates into

$$\int_{-\infty}^{\infty} d\nu\,[\Phi(r+\nu\,n),\Phi(q-r-\nu\,n)]=0$$

Where *n* is a timelike 4-vector, but *r* and *q* are arbitrary, apart from the condition $r.n = 0$.

This can actually be solved for interacting fields as well, order-by-order in the coupling constant. Haag (1955) did the calculation up to first order and found that a local field equation with a possible derivative coupling solves this. The speaker solved up to infinite order, but with the additional requirement that the time derivatives of the fields also commute with the field and with each other, also finding local field equations as the solution, but with derivative couplings not allowed. The indications seemed to be that local field equations *with the normal-ordering modifications needed to avoid infinities*, also solved the commutators.

Quantum electrodynamics and beyond

The scalar field theory described here was for illustration only. Almost all the work has been in the domain of Quantum Electrodynamics. One finds that

1. The photon mass can be arbitrarily small, but cannot be zero
2. Scattering amplitudes for QED up to tree level agree with Feynman graph analysis.
3. A chiral version of the theory seems to require self-interactions between vector bosons – maybe there is a Higgs-less Standard Model here - ?

Replicating the “tests” of quantum electrodynamics requires an understanding of bound states for the Lamb Shift and the classical limit of the photon field for the anomalous magnetic moment.

Theoretically, SCPT should allow treatment of processes other than scattering as the time variable is not eliminated. Theoretically ... but it remains elusive!