STÜCKELBERG COVARIANT PERTURBATION THEORY IN 5 MINUTES

C.G. Oakley, 17 February 2012 (last modified 2 July 2015)

The equation of motion of Φ^3 scalar field theory is

$$(\partial^2 + m^2)\Phi(x) = -\lambda\Phi^2(x) \tag{1}$$

The standard technique is to create an interaction picture time-evolution operator that acts on quasi-free field states identified as incoming particles in a scattering experiment, and to work out the matrix element with quasi-free field states identified as outgoing particles. Stückelberg's method involves none of this. Instead one expresses the interacting field as a Maclaurin series in the coupling constant:

$$\Phi(x) = \Phi_0(x) + \lambda\Phi_1(x) + \lambda^2\Phi_2(x) + \cdots \tag{2}$$

When this is substituted back into (1) one gets an infinite number of equations as follows:

$$\begin{aligned}
(\partial^2 + m^2)\Phi_0(x) &= 0\\
(\partial^2 + m^2)\Phi_1(x) &= -\Phi_0(x)^2\\
(\partial^2 + m^2)\Phi_2(x) &= -(\Phi_0(x)\Phi_1(x) + \Phi_1(x)\Phi_0(x))\\
&\vdots\\
(\partial^2 + m^2)\Phi_r(x) &= -\sum_{i=0}^{r-1}\Phi_i(x)\Phi_{r-i-1}(x)
\end{aligned} \tag{3}$$

Evidently $\Phi_0(x)$ is a free field, and the higher order Φ's are directly or indirectly determined from this, although each is arbitrary to the extent of addition of a free-field solution. Note that Φ_0 is a formal construct here—no attempt is being made to identify it with "in" or "out" scattering states.

The expansion can also be substituted into the equal-time commutators

$$\begin{aligned}
[\Phi(\mathbf{x}, t), \dot{\Phi}(\mathbf{x}', t)] &= i\delta(\mathbf{x} - \mathbf{x}')\\
[\Phi(\mathbf{x}, t), \Phi(\mathbf{x}', t)] &= 0
\end{aligned} \tag{4}$$

from which we find that Φ_0 has the standard equal-time commutator. Higher orders can be solved—Haag attempted something like this in his 1955 paper, inserting an expansion in terms of free fields with no assumptions about the interacting equation of motion, and the indications were that local field equations, such as Φ^3 theory here were what was needed to solve to all orders.

The negative-energy part of Φ_0 must annihilate the vacuum, even in the presence of interactions, since we would otherwise have negative-energy states.

These simple things, namely the reduction of the interacting field in terms of Φ_0, the knowledge that Φ_0 has the expected commutator function and the knowledge that its negative-energy part annihilates the vacuum, give one a framework for calculating matrix elements.

One finds, from second-order upwards, pathological infinities arising when one commutes the negative-energy parts past to annihilate the vacuum.

If, on the other hand, one's starting point was

$$(\partial^2 + m^2)\Phi(x) = -\lambda : \Phi^2(x) : \tag{5}$$

instead of (1) (i.e. one was allowed to place the terms from lower order in normal order), then these would not appear: there are still loop integrals, but these do not diverge.

Whether the normal-ordered version still solves the commutators (4) is not clear, but it seems likely.

Being able to calculate matrix elements is not, however, the end of the story because of the disquieting property that

$$\langle 0|\Phi(t, \mathbf{x})\Phi(t, \mathbf{x}')\Phi(t, \mathbf{x}'')|0\rangle \neq 0 \tag{6}$$

In other words, a two-particle state at time t is not orthogonal to a one-particle state at the same time! This, in fact, is a direct consequence of Haag's theorem, which, crudely stated, says that there is no interaction picture (if there was, we could unitarily transform the fields in (6) into free fields, whereupon the matrix element would vanish identically).

It would appear, though, that despite this, the undesired behaviour takes the form of transients. The loop integrals seem also to lead just to transients—the contributions that survive for large time elapse between initial and final states can be identified by poles in the amplitude, and these give the same expressions as tree-level Feynman graphs.

Note: no interaction picture and no time-ordered products. The Heisenberg picture is maintained throughout.

Actually, a normal-ordered power-series expansion in the coupling may not be the answer. Here is what I am investigating at present (July 2015). I will post more details when I have the time, but these are the essentials: in the non-relativistic limit of (SCPT) quantum electrodynamics the Bethe-Salpeter equation, which is the self-consistency equation for the sum of multi-photon-exchange ladder graphs for a 2-fermion bound state, correctly gives the gross structure (it may well give the fine structure and Lamb shift, too—I am still working on that). Obviously these results are all finite and require no regularization/renormalization, and yet if one calculates the two-photon-exchange ladder graph, which is a term in this series, it diverges! So the sum of the series is finite, but individual terms diverge, suggesting that the expansion itself is at fault. Is there a way of avoiding expansion in the coupling, I wonder? The self-consistency equation can certainly be determined just by looking at the raw Q.E.D. equations of motion, and making a few approximations, but what is really needed is an alternative expansion in which this self-consistency relation appears as a term.