

The two-body relativistic bound state

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We start with the equations of motion of quantum electrodynamics

$$(i\not{\partial} - e\not{A} - m)\psi(x) = 0 \quad (1)$$

$$(\partial^2 + \mu^2)A^\mu(x) = e\bar{\psi}(x)\gamma^\mu\psi(x) \quad (2)$$

where we allow for a possible photon mass. In terms of Fourier transforms defined by

$$\psi(p) = (2\pi)^{-4} \int d^4x e^{-ip\cdot x} \psi(x) \quad (3)$$

and similarly for A_μ , these equations become

$$(\not{p} + m)\psi(p) = -e \int d^4q \not{A}(q)\psi(p - q) \quad (4)$$

$$(p^2 - \mu^2)A^\mu(p) = -e \int d^4q \bar{\psi}(q)\gamma^\mu\psi(p + q) \quad (5)$$

Here we consider a particle-antiparticle system of two non-identical fundamental fermions ψ and χ with the same charge. Combining (4), and (5) for the χ species, and considering just the χ current, we have

$$\psi_\alpha(p) = - \left(\frac{e}{\not{p} + m} \right)_\alpha^\beta \int d^4q \frac{-e}{q^2 - \mu^2} \int d^4q' (\bar{\chi}_0(q')\gamma^\mu\chi_0(q + q')) (\gamma_\mu\psi(p - q))_\beta \quad (6)$$

where we are using early Greek letters for Dirac spinor indices. We have approximated χ with its free-field value χ_0 . Now apply $\bar{\chi}_0^\beta(-p_1)|\rangle$ to the right, where $|\rangle$ is the vacuum state. This gives

$$\psi_\alpha(p_2)\bar{\chi}_0^\beta(-p_1)|\rangle = \left(\frac{e^2}{\not{p}_2 + m} \right)_\alpha^\gamma \int \frac{d^4q}{q^2 - \mu^2} d^4q' (\bar{\chi}_0(q')\gamma^\mu\chi_0(q + q')) (\gamma_\mu\psi(p_2 - q))_\gamma \bar{\chi}_0^\beta(-p_1)|\rangle \quad (7)$$

χ_0 can be anticommutated past ψ and then $\bar{\chi}_0$ to annihilate the vacuum. We then pick up the positive-energy part of the anticommutator function

$$\{\chi_{0\alpha}(p), \bar{\chi}_0^\beta(q)\} = (\not{p} - M)_\alpha^\beta \delta(p - q) \delta(p^2 - M^2) \epsilon(p_0) \quad (8)$$

where M is the χ fermion mass. This gives

$$\psi_\alpha(p_2)\bar{\chi}_0^\beta(-p_1)|\rangle = e^2 (\not{p}_2 + m)^{-1} \alpha^\gamma \int \frac{d^4q}{q^2 - \mu^2} (\bar{\chi}_0(-q - p_1)\gamma^\mu)^\delta (\gamma_\mu\psi(p_2 - q))_\gamma (-\not{p}_1 - M)_\delta^\beta \delta(p_1^2 - M^2) \theta(p_1^0)|\rangle \quad (9)$$

Thus

$$\psi_\alpha(p_2)\bar{\chi}_0^\beta(-p_1)|\rangle = e^2 \delta(p_1^2 - M^2) \theta(p_1^0) (\gamma^\mu(\not{p}_1 + M))_\delta^\beta ((\not{p}_2 + m)^{-1} \gamma_\mu)_\alpha^\gamma \int \frac{d^4q}{q^2 - \mu^2} \psi_\gamma(p_2 - q) \bar{\chi}_0^\delta(-q - p_1)|\rangle \quad (10)$$

Write $p_2 = p - p_1$. Then

$$\psi_\alpha(p - p_1)\bar{\chi}_0^\beta(-p_1)|\rangle = e^2 \delta(p_1^2 - M^2) \theta(p_1^0) (\gamma^\mu(\not{p}_1 + M))_\delta^\beta ((\not{p} - \not{p}_1 + m)^{-1} \gamma_\mu)_\alpha^\gamma \int \frac{d^4q}{(q - p_1)^2 - \mu^2} \psi_\gamma(p - q) \bar{\chi}_0^\delta(-q)|\rangle \quad (11)$$

Multiplying through by $\not{p} - \not{p}_1 + m$, we have

$$(\not{p} - \not{p}_1 + m)_\alpha^\gamma \psi_\gamma(p - p_1) \bar{\chi}_0^\beta(-p_1) = e^2 \delta(p_1^2 - M^2) \theta(p_1^0) \int \frac{d^4 q}{(q - p_1)^2 - \mu^2} (\gamma_\mu)_\alpha^\gamma \psi_\gamma(p - q) \bar{\chi}_0^\delta(-q) (\gamma^\mu (\not{p}_1 + M))_\delta^\beta \quad (12)$$

We then write

$$\psi_\alpha(p - p_1) \bar{\chi}_0^\beta(-p_1) = g_\alpha^\beta(p, p_1) \delta(p_1^2 - M^2) \theta(p_1^0) \quad (13)$$

From which we find

$$(\not{p} - \not{p}_1 + m)g(p, p_1) = e^2 \int \frac{d^4 q \theta(q_0) \delta(q^2 - M^2)}{(q - p_1)^2 - \mu^2} \gamma^\mu g(p, q) \gamma_\mu (\not{p}_1 + M) \quad (14)$$

where we have suppressed the Dirac spinor indices. This is similar, if not actually identical to Cowen-Greenberg equation 30. One of the differences is conventions: we must replace e^2 with $e^2/(2\pi)^3$ to compare. This factor arises because equation 8 here conventionally has an extra $(2\pi)^3$.

Here we attempt to solve the equation in the chiral representation, where we explicitly show the Dirac spinor as a direct sum of two-component $\text{SL}(2, \mathbb{C})$ objects:

$$\psi_\alpha(p) = \begin{bmatrix} \psi_A(p) \\ \bar{\psi}^{A'}(-p) \end{bmatrix}; \quad \text{and} \quad \bar{\psi}^\beta(-p) = \begin{bmatrix} \tilde{\psi}^B(p) & \bar{\psi}_{B'}(-p) \end{bmatrix} \quad (15)$$

and similarly for χ_0 . In the chiral representation, with Penrose conventions for $\text{SL}(2, \mathbb{C})$ we have

$$\gamma^\mu = \sqrt{2} \begin{bmatrix} 0 & \sigma_{BA'}^\mu \\ \sigma^{\mu AB'} & 0 \end{bmatrix} \quad (16)$$

Since χ_0 satisfies the free Dirac equation $\bar{\chi}_0(-p_1)(-\not{p}_1 + M) = 0$, we can write

$$\bar{\chi}_0^\beta(-p) = \begin{bmatrix} \tilde{\chi}_0^B(p) & \frac{\sqrt{2}}{M} p_{AB'} \tilde{\chi}_0^A(p) \end{bmatrix} \quad (17)$$

Substituting into (12), we find just one independent equation for the β index, which is

$$[(\not{p} - \not{p}_1 + m)\psi(p - p_1)]_\alpha \tilde{\chi}_0^\beta(p_1) = 2e^2 \delta(p_1^2 - M^2) \theta(p_1^0) \int \frac{d^4 q}{(q - p_1)^2 - \mu^2} [\gamma_\mu \psi(p - q)]_\alpha (\sigma^{\mu BC'} q_{CC'} + p_1^{BC'} \sigma_{CC'}^\mu) \tilde{\chi}_0^C(q) \quad (18)$$

We may now write out ψ in terms of two components as well. This gives

$$\begin{bmatrix} m\psi_A(p - p_1) + \sqrt{2}(p - p_1)_{AB'} \bar{\psi}^{B'}(-p + p_1) \\ \sqrt{2}(p - p_1)^{DA'} \psi_D(p - p_1) + m\bar{\psi}^{A'}(-p + p_1) \end{bmatrix} \tilde{\chi}_0^B(p_1) = 2\sqrt{2}e^2 \delta(p_1^2 - M^2) \theta(p_1^0) \int \frac{d^4 q}{(q - p_1)^2 - \mu^2} \begin{bmatrix} (\delta_A^B q_{CB'} - \epsilon_{AC} p_1^{B'}) \bar{\psi}^{B'}(-p + q) \\ (\epsilon^{DB} q_{CA'} + \delta_C^D p_1^{BA'}) \psi_D(p - q) \end{bmatrix} \tilde{\chi}_0^C(q) \quad (19)$$

Solving for $\bar{\psi}^{B'} \tilde{\chi}_0^C$ in the bottom equation, and resubstituting into the top equation, we get

$$((p - p_1)^2 - m^2) \psi_A(p - p_1) \tilde{\chi}'_{0B}(p_1) = 4e^2 \int \frac{dM(q)}{(q - p_1)^2 - \mu^2} R_{AB}^{CD}(p, p_1, q) \psi_C(p - q) \tilde{\chi}'_{0D}(q) - 8e^4 \int \frac{dM(q)}{(q - p_1)^2 - \mu^2} \frac{dM(q')}{(q - q')^2 - \mu^2} S_{AB}^{CD}(p, q, q') \psi_C(p - q') \tilde{\chi}'_{0D}(q') \quad (20)$$

Here we have written

$$\chi_{0B}(p_1) = \theta(p_1^0)\delta(p_1^2 - M^2)\tilde{\chi}'_{0B}(p_1) \quad (21)$$

which is possible since χ_0 obeys the Klein-Gordon equation. We have used the shorthand $dM(q)$ for the invariant measure $d^4q\theta(q_0)\delta(q^2 - M^2)$. The tensors R_{AB}^{CD} and S_{AB}^{CD} are defined as follows:

$$R_{AB}^{CD} = \delta_{AB}^{CD}(\frac{1}{2}(p_1 + q) \cdot p - M^2) + \frac{1}{2}\delta_{AB}^{DC}(p_1 - q)^2 + \delta_B^D I(p, p_1 - q)_A^C + \delta_A^D I(p_1 - q, p - q)_B^C + \delta_B^C I(p - p_1, q - p_1)_A^D \quad (22)$$

and

$$S_{AB}^{CD}(p_1, q, q') = \delta_{AB}^{CD}(M^2 + \frac{1}{2}(p_1 \cdot (q' - q) - q \cdot q')) + \delta_{AB}^{DC}(\frac{1}{2}q \cdot (p_1 + q') - M^2) + \delta_A^C I(q - p_1, q - q')_B^D - \delta_A^D I(q - p_1, q)_B^C - \delta_B^C I(q, q - q')_A^D \quad (23)$$

$I(p, q)$ is defined as:

$$I(p, q)_A^B = p_{AA'}q^{BA'} - \frac{1}{2}\delta_A^B(p \cdot q) \quad (24)$$

This is traceless, linear in p and q and has the property that $I(p, p) = 0$. Hence $I(p, q) = -I(q, p)$. It can be written as

$$I(p, q)_A^B = \frac{1}{2}(q^0 \mathbf{p} - p^0 \mathbf{q} - i\mathbf{p} \wedge \mathbf{q}) \cdot \tau_A^B \quad (25)$$

where τ are the Pauli matrices.

To associate the states with a time, we Fourier transform back just the time component of the fields. Thus

$$|t; \mathbf{p}; \mathbf{p}_1; AB\rangle = \psi_A(t; \mathbf{p} - \mathbf{p}_1)\tilde{\chi}_{0B}(t; \mathbf{p}_1)|\rangle = \int dp_0 dp_1^0 e^{ip_0 t} \psi_A(p - p_1)\tilde{\chi}_{0B}(p_1)|\rangle \Big|_{p_1 \text{ on shell}} \quad (26)$$

For the purpose of calculating the inner product, we approximate ψ with its free-field value, giving

$$\langle t'; \mathbf{p}'; \mathbf{p}'_1; A'B' | t; \mathbf{p}; \mathbf{p}_1; AB \rangle = \delta(\mathbf{p} - \mathbf{p}')\delta(\mathbf{p}_1 - \mathbf{p}'_1) \frac{e^{i(\sqrt{\mathbf{p}_1^2 + M^2} + \sqrt{(\mathbf{p} - \mathbf{p}_1)^2 + m^2})(t - t')}}{2\sqrt{\mathbf{p}_1^2 + M^2}\sqrt{(\mathbf{p} - \mathbf{p}_1)^2 + m^2}} (p - p_1)_{AA'} p_{1BB'} \quad (27)$$

Thus the states

$$|t; \mathbf{p}; \mathbf{p}_1; ij\rangle = \frac{2}{\sqrt{mM}} \left(\frac{\mathbf{p}_1^2 + M^2}{(\mathbf{p} - \mathbf{p}_1)^2 + m^2} \right)^{\frac{1}{4}} L_i^A(p - p_1) L_j^B(p_1) |t; \mathbf{p}; \mathbf{p}_1; AB\rangle \quad (28)$$

are orthonormal, i.e.

$$\langle t; \mathbf{p}'; \mathbf{p}'_1; kl | t; \mathbf{p}; \mathbf{p}_1; ij \rangle = \delta(\mathbf{p} - \mathbf{p}')\delta(\mathbf{p}_1 - \mathbf{p}'_1)\delta_{ik}\delta_{jl} \quad (29)$$

where

$$L_i^A(q) = \frac{1}{\sqrt{2m_0(q_0 + m_0)}} (q_0 + m_0 + \mathbf{q} \cdot \tau)_i^A \quad (30)$$

with $q^2 = m_0^2$, is the boost that transforms to a frame where q is at rest. Thus

$$\begin{aligned} & ((p - p_1)^2 - m^2)(\mathbf{p}_1^2 + M^2)^{\frac{1}{4}}((\mathbf{p} - \mathbf{p}_1)^2 + m^2)^{\frac{1}{4}} L^{-1}_A{}^i(p - p_1) L^{-1}_B{}^j(p_1) |t; p; p_1; ij\rangle = \\ & 4e^2 \int \frac{dM(q)}{(q - p_1)^2 - \mu^2} R_{AB}^{CD}(p, p_1, q) (\mathbf{q}^2 + M^2)^{\frac{1}{4}} ((\mathbf{p} - \mathbf{q})^2 + m^2)^{\frac{1}{4}} L_C^{-1k}(p - q) L^{-1}_D{}^l(q) |t; p; q; kl\rangle - \\ & 8e^4 \int \frac{dM(q)}{(q - p_1)^2 - \mu^2} \frac{dM(q')}{(q - q')^2 - \mu^2} S_{AB}^{CD}(p, q, q') (\mathbf{q}'^2 + M^2)^{\frac{1}{4}} ((\mathbf{p} - \mathbf{q}')^2 - m^2)^{\frac{1}{4}} \\ & L^{-1}_C{}^k(p - q') L^{-1}_D{}^l(q) |t; p; q'; kl\rangle \quad (31) \end{aligned}$$

where $p_1^0 = \sqrt{\mathbf{p}_1^2 + M^2}$ and p_0 is now the operator $-i\partial/\partial t$.

We now consider the case of non-relativistic motion. Put $\mathbf{p} = 0$ and assume that $\mathbf{p}_1 \ll M$. Also, ignore the e^4 contribution. The Lorentz transformations here are then approximately the identity and R_{AB}^{CD} approximates to $mM\delta_{AB}^{CD}$:

$$\left(\left(p^0 - \sqrt{\mathbf{p}_1^2 + M^2} \right)^2 - \mathbf{p}_1^2 - m^2 \right) |t; p; p_{1;ij} \rangle \approx 4e^2 \int \frac{d^3\mathbf{q}}{2\sqrt{\mathbf{q}^2 + M^2}((q - p_1)^2 - \mu^2)} mM |t; p; q_{ij} \rangle \quad (32)$$

Now

$$(q - p_1)^2 = \left(\sqrt{\mathbf{q}^2 + M^2} - \sqrt{\mathbf{p}_1^2 + M^2} \right)^2 - (\mathbf{q} - \mathbf{p}_1)^2 \approx \left(\frac{\mathbf{q}^2}{2M} - \frac{\mathbf{p}_1^2}{2M} \right)^2 - (\mathbf{q} - \mathbf{p}_1)^2 \approx -(\mathbf{q} - \mathbf{p}_1)^2 \quad (33)$$

So

$$\begin{aligned} \left(p^0 - \sqrt{\mathbf{p}_1^2 + M^2} - \sqrt{\mathbf{p}_1^2 + m^2} \right) \left(p^0 - \sqrt{\mathbf{p}_1^2 + M^2} + \sqrt{\mathbf{p}_1^2 + m^2} \right) |t; p; p_{1;ij} \rangle \approx \\ -2e^2 m \int \frac{d^3\mathbf{q}}{(\mathbf{q} - \mathbf{p}_1)^2 + \mu^2} |t; p; q_{ij} \rangle \end{aligned} \quad (34)$$

So, at our level of approximation

$$\left(p^0 - m - M - \frac{\mathbf{p}_1^2}{2m_r} \right) |t; p; p_{1;ij} \rangle = -e^2 \int \frac{d^3\mathbf{q}}{(\mathbf{q} - \mathbf{p}_1)^2 + \mu^2} |t; p; q_{ij} \rangle \quad (35)$$

where $m_r = mM/(m + M)$ is the reduced mass.

We may use the fact that $-i\hat{\mathbf{x}}$ is the momentum displacement operator on account of the fact that $[x_i, p_j] = i\delta_{ij}$. Thus

$$|\mathbf{q}\rangle = e^{-i(\mathbf{q}-\mathbf{p}_1)\cdot\hat{\mathbf{x}}} |\mathbf{p}_1\rangle \quad (36)$$

Hence

$$\left(p^0 - m - M - \frac{\mathbf{p}_1^2}{2m_r} + e^2 \int \frac{d^3\mathbf{q}}{(\mathbf{q} - \mathbf{p}_1)^2 + \mu^2} e^{-i(\mathbf{q}-\mathbf{p}_1)\cdot\hat{\mathbf{x}}} \right) |t; p; p_{1;ij} \rangle = 0 \quad (37)$$

Now

$$\int d^3\mathbf{q} \frac{e^{-i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 + \mu^2} = \frac{2\pi^2}{|\mathbf{x}|} e^{-\mu|\mathbf{x}|} \quad (38)$$

Hence

$$\left(p^0 - m - M - \frac{\mathbf{p}_1^2}{2m_r} + e^2 \frac{2\pi^2}{|\hat{\mathbf{x}}|} e^{-\mu|\hat{\mathbf{x}}|} \right) |t; p; p_{1;ij} \rangle = 0 \quad (39)$$

With the conventions here, the fine-structure constant is defined by $\alpha = 2\pi^2 e^2$, giving us

$$\left(p^0 - m - M - \frac{\mathbf{p}_1^2}{2m_r} + \frac{\alpha}{|\hat{\mathbf{x}}|} e^{-\mu|\hat{\mathbf{x}}|} \right) |t; p; p_{1;ij} \rangle = 0 \quad (40)$$

If we put the photon mass μ to zero, this then leads to the usual gross structure of a single-electron atom.